Conformally symmetric circle packings. A generalization of Doyle spirals

Alexander I. Bobenko[1] and Tim Hoffmann[2]

Fachbereich Mathematik, Technische Universität Berlin,
Straße des 17 Juni 136, 10623 Berlin, Germany

1 Introduction

Circle packings (and more generally patterns) as discrete analogs of conformal mappings is a fast developing field of research on the border of analysis and geometry. Recent progress was initiated by Thurston's idea [T] about the approximation of the Riemann mapping by circle packings. The corresponding convergence was proven by Rodin and Sullivan [RS]; many additional connections with analytic functions, such as the discrete maximum principle and Schwarz's lemma [R], the discrete uniformization theorem [BS], etc., have emerged since then.

The topic "circle packings" is also a natural one for computer experimentation and visualization. Computer experiments demonstrate a surprisingly close analogy of the classical theory in the emerging "discrete analytic function theory" [DS]. Although computer experiments give convincing evidence for the existence of discrete analogs of many standard holomorphic functions, the Doyle spirals (which are discrete analogs of the exponential function, see section 4) are the only circle packings described explicitly.

Circle packings are usually described analytically in the Euclidean setting, i.e. through their radii function. On the other hand, circles and the tangencies are preserved by the fractional-linear transformations of the Riemann sphere (Möbius transformations). It is natural to study circle packings in this setting, i.e. modulo the group of the Möbius transformations. He and Schramm [HS] developed a conformal description of hexagonal circle packings, which helped them to show that Thurston's convergence of hexagonal circle packings to the Riemann mapping is actually C^∞. They describe circle packings in terms of the cross-ratios

$$q(a,b,c,d) := \frac{(a-b)(c-d)}{(b-c)(d-a)}$$

of their touching points.

In [S] Schramm introduced circle patterns with the combinatorics of the square grid (SG patterns). In many aspects the SG theory is analogous to the theory of the hexagonal circle packings.

[1]E–mail: `bobenko @ math.tu-berlin.de`
[2]E–mail: `timh @ sfb288.math.tu-berlin.de`

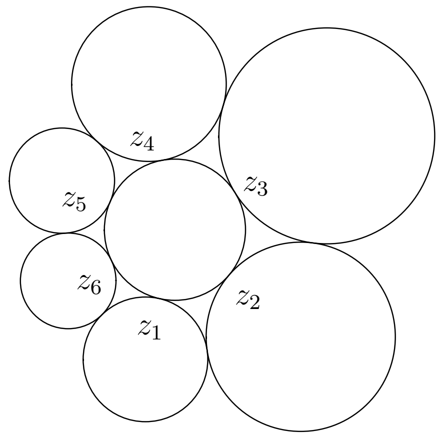

Figure 1: A circle flower.

However, the SG theory is analytically simpler. The corresponding discrete equations describing the SG patterns, in the Euclidean as well as in the conformal setting, turn out to be integrable [BP]. Methods of the theory of integrable equations allowed us in [3] [AB] to find Schramm's circle patterns which are analogs of the holomorphic functions $z^\alpha, \log z$.

One big question is which results on the Schramm's circle patterns carry over to the hexagonal setting, in particular whether some discrete standard functions can be described explicitly. This is closely related to the question of integrability of the basic discrete equations for hexagonal circle packings (He-Schramm equation, see section 3). In the present paper the first simple step in this direction is made. We study (surprisingly non-trivial) conformal geometry of hexagonal circle packings. In terms of this approach, a special class of *conformally symmetric* circle packings, which are generalizations of the Doyle spirals, is introduced and all such packings are described explicitly.

Since this paper deals with families of circle packings it seems natural to show not only arbitrarily choosen members in the figures, but to provide a possibility to present them all. Therefore there is an interactive version of this paper available [online]. It has some of the figures replaced by applets, that allow to explore the families directly. See section 6 for more information on this version.

Acknowledgments. The authors thank U. Hertrich-Jeromin, U. Pinkall, Yu. Suris and E. Tjaden for helpful discussions.

[3]Discrete z^2 and $\log z$ have been conjectured by Schramm and Kenyon earlier [Swww].

2 Geometry of circle flowers and conformally symmetric circle packings

This paper concerns patterns of circles in the plane called hexagonal circle packings. Their basic unit is the *flower*, consisting of a *center* circle tangent to and surrounded by *petals*. A *hexagonal* flower is illustrated in Fig.1; the six petals form a closed chain which wraps once in the positive direction about the center. Whereas the neighboring petals touch, the circles of not-neighboring petals of a flower may intersect. We call a flower *immersed* if none of its circles degenerates to a point. A hexagonal circle packing is a collection of oriented circles where each of its internal circles is the center of a hexagonal flower. Orientations of the circles should agree: at the touching points the orientations of the touching circles must be opposite. A hexagonal circle packing can be labeled by the triangular (hexagonal) lattice

$$HL = n + me^{i\pi/3} \in \mathbb{C}, \qquad n, m \in \mathbb{Z}.$$

or by one of its subset. A circle packing is called *immersed* if all its flowers are immersed. Immersions of the whole HL are called *entire*. Fractional-linear transformations of the complex plane (Möbius transformations) preserve circles, their orientation and their tangencies. In this paper we study circle packings modulo the group of Möbius transformations.

The center circle of a flower contains 6 points $z_1, \ldots, z_6$ (see Fig.1) where it touches the petals. We call them *center touching points* of a circle flower.

Proposition 2.1 *Let $z_1, \ldots, z_6$ be cyclicly ordered*[4] *points on a circle C. Then the following three statements are equivalent:*

(i) There exists a flower with the center C and center touching points $z_1, \ldots, z_6$,

(ii) The multi-ratio m of $z_1, \ldots, z_6$ is equal to -1, i.e.

$$m(z_1, z_2, z_3, z_4, z_5, z_6) := \frac{(z_1 - z_2)(z_3 - z_4)(z_5 - z_6)}{(z_2 - z_3)(z_4 - z_5)(z_6 - z_1)} = -1, \tag{1}$$

(iii) There exists an involutive Möbius transformation M (Möbius involution) such that

$$M(z_k) = z_{k+3} \qquad (k \bmod 6).$$

Proof. Mapping the point z_6 to infinity by a Möbius transformation one obtains two parallel lines and five touching circles as in Fig.2. An elementary computation yields

$$z_{k+1} - z_k = 2\sqrt{r_{k+1} r_k}, \qquad k = 1, \ldots, 4, \tag{2}$$

where r_k are the radii of the corresponding circles. Together with $r_1 = r_5$ and $(z_5 - z_6)/(z_6 - z_1) = -1$ this implies (1).

[4]Positive orientation of the ordering is assumed.

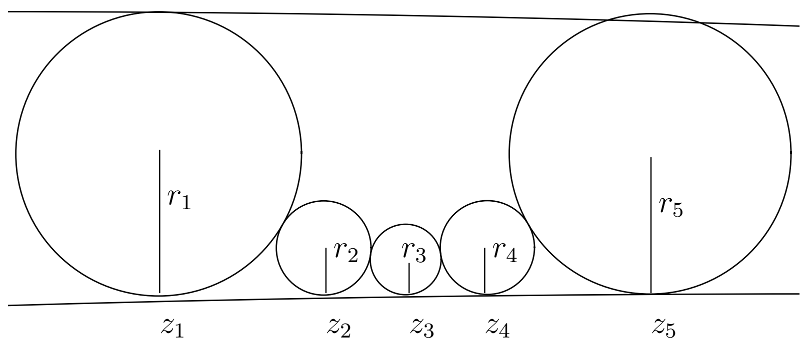

Figure 2: A flower with one central touching point at infinity.

On the other hand, given arbitrary $r_1 > 0$ and ordered $z_1, \dots, z_6$ satisfying (1), after normalizing $z_6 = \infty$ formula (2) provides us with the radii of the touching circles as in Fig.2. This proves the equivalence of (i) and (ii).

To show the equivalence of (ii) and (iii), define the Möbius transformation M through $M(z_1) = z_4, M(z_2) = z_5, M(z_3) = z_6$. Consider $z_* = M(z_4)$. The invariance of the cross-ratios $q(z_1, z_2, z_3, z_4) = q(z_4, z_5, z_6, z_*)$ implies the equivalence of (1) and $z_* = z_1$. The same proof holds for $M(z_5) = z_2$ and $M(z_6) = z_1$.

To each center touching point z_k of a flower, one can associate a circle S_k passing through 4 touching points $z_{k-1}, z_{k+1}, w_k, w_{k-1}$ of the flower containing z_k (see Fig.3). Here w_k is the touching point of petals[5] $\mathcal{P}_{k+1}$ and $\mathcal{P}_k$. Indeed, mapping the point z_k by a Möbius transformation to ∞, it is easy to see that the points $z_{k-1}, z_{k+1}, w_k, w_{k-1}$ are mapped to vertices of a rectangle, thus lie on a circle. We call these circles *s-circles* of a flower.

Theorem 2.2 *There exist a one-parameter family of flowers with the same center touching points. Moreover, there exists a unique flower F in this family, which satisfies the following equivalent conditions:*

(i) F is invariant with respect to a Möbius involution M with a fixed point P,

(ii) All s-circles of F intersect in one point P.

We call the flower F of the theorem *conformally symmetric.*

One can view the whole family of flowers at [appl].

Proof. Keeping the points $z_1, \dots, z_5$ in Fig.2 fixed and varying r_1 one obtains a one parameter family of flowers with the same touching central points. Let us now construct the flower F. The Möbius involution of Proposition 2.1 preserves the central circle C. Consider the circles

[5]The petals are labeled by the corresponding touching points z_k.

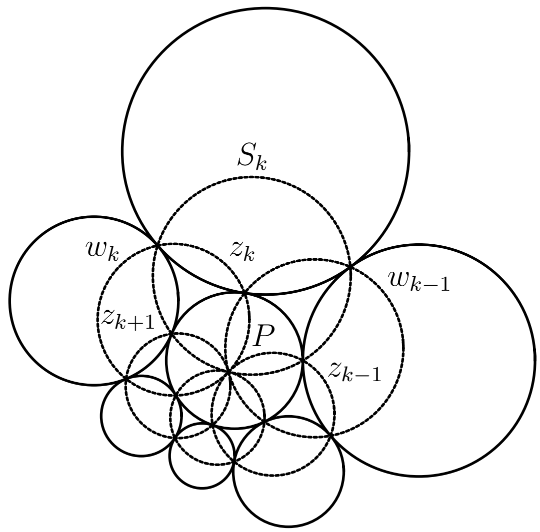

Figure 3: A conformally symmetric flower.

$C_k,\ i=1,2,3$, orthogonal to C and passing through the pairs of points $\{z_k, z_{k+3}\}$. All these three circles intersect in 2 points P and P', which are the fixed points of M lying inside and outside C, respectively. By a Möbius transformation, map the point P to infinity. The Möbius involution M becomes $M(z) = -z$ and the circles C_1, C_2, C_3 become straight lines intersecting in the center of C. To construct the flower F, connect the z_k-points with even (respectively, with odd) labels by straight lines and consider their intersection points w_k (see Fig.4). The circles C_k passing through the triples w_k, w_{k-1}, z_k touch at the points w_k. Let us prove this fact for C_1 and C_2. Indeed, the triangles $\Delta(w_1, w_6, z_1)$ and $\Delta(z_3, z_5, z_1)$ are similar, therefore the tangent lines to the circle C_1 at w_1 and to the circle C at z_3 are parallel. The tangent lines to C_2 at w_1 and to C at z_6 are also parallel. Since the points z_3 and z_6 are opposite on C, the circles C_1 and C_2 touch at w_1. The circles C_k are the petals of the desired flower F, which is obviously M-symmetric. The s-circles of this flower are the straight lines (z_k, z_{k+2}). The latter obviously intersect at infinity, thus all the s-circles of F intersect in the fixed point P of M.

The proof of $(ii) \Rightarrow (i)$ is similar. After mapping the point P to infinity the s-circles become straight lines and the flower is as in Fig.4. Since the circles in this figure touch, their tangent lines at the points z_k, z_{k+3} and w_{k+1} are parallel. This implies that z_k and z_{k+3} are opposite points on C, and the flower is symmetric with respect to the π-rotation of C.

Definition 2.3 *A hexagonal circle packing is called* conformally symmetric *or an s-circle packing if it consists of conformally symmetric flowers, i.e. the s-circles of each of its flowers intersect in one point.*

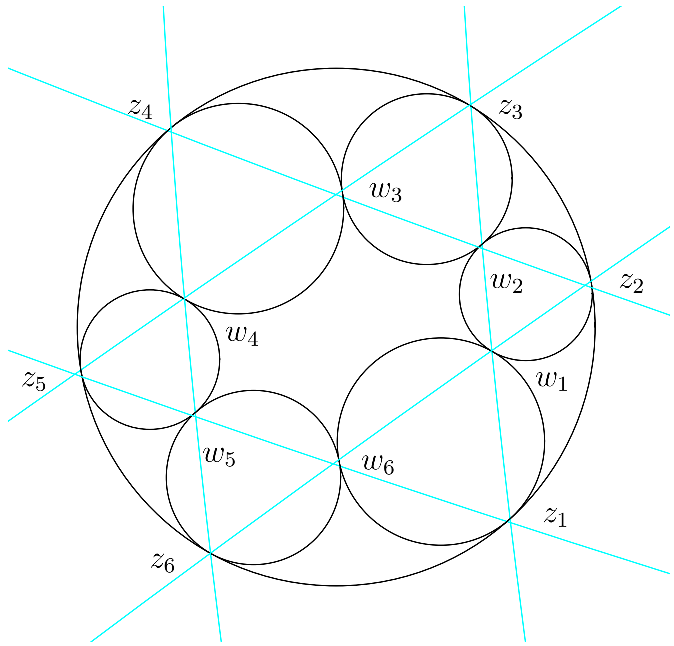

Figure 4: A normalized conformally symmetric flower.

3 Analytic description of conformally symmetric circle packings

In this section we describe all conformally symmetric circle packings using the conformal description of circle packings proposed by He and Schramm [HS].

To each central touching point z_k of a flower one associates the cross-ratio[6]

$$s_k := q(z_k, z_{k-1}, w_{k-1}, w_k) = \frac{(z_k - z_{k-1})(w_{k-1} - w_k)}{(z_{k-1} - w_{k-1})(w_k - z_k)}. \tag{3}$$

Mapping z_k to ∞, one observes that tree other points in (3) are mapped to vertices of a rectangle, which implies that s_k is purely imaginary. Moreover, the cross-ratios of an immersed oriented flower are positive imaginary, $-is_k > 0$. Also note that

$$s_k = -q(z_{k+1}, z_{k-1}, w_{k-1}, z_k) = q(z_k, w_k, z_{k+1}, z_{k-1}), \tag{4}$$

and that $s_k^2 = q(z_{k+1}, z_{k-1}, w_{k-1}, w_k)$ is the cross-ratio of the four touching points lying on the s-circle S_k.

Lemma 3.1 *The cross-ratios s_k of a flower satisfy the He-Schramm equation [HS]*

$$s_k + s_{k+2} + s_{k+4} + s_k s_{k+1} s_{k+2} = 0 \tag{5}$$

[6] Note that our normalization of s_k differs from the one in [HS].

for all k mod 6.

Proof. Let m_k be the Möbius transformation that takes z_k, z_{k-1}, w_{k-1} to the points $\infty, 0, 1$, respectively. By the definition of s_k we have

$$\begin{aligned} s_k &= q(z_k, z_{k-1}, w_{k-1}, w_k) = q(\infty, 0, 1, m_k(w_k)), \\ -s_k &= q(z_{k+1}, z_{k-1}, w_{k-1}, z_k) = q(m_k(z_{k+1}), 0, 1, \infty), \end{aligned}$$

thus

$$m_k(w_k) = 1 - s_k, \qquad m_k(z_{k+1}) = -s_k.$$

For $M_k := m_{k+1} m_1^{-1}$ this yields $M_k(-s_k) = \infty, M_k(\infty) = 0, M_k(1 - s_k) = 1$ and, finally,

$$M_k = \begin{pmatrix} 0 & 1 \\ 1 & s_k \end{pmatrix},$$

where the usual matrix notation for the Möbius transformation is used. The equality of the corresponding Möbius transformations implies $M_3 M_2 M_1 = \pm M_4^{-1} M_5^{-1} M_6^{-1}$, which is

$$\begin{pmatrix} s_2 & 1 + s_1 s_2 \\ 1 + s_2 s_3 & s_1 + s_3 + s_1 s_2 s_3 \end{pmatrix} = \pm \begin{pmatrix} -s_4 - s_6 - s_4 s_5 s_6 & 1 + s_4 s_5 \\ 1 + s_5 s_6 & -s_5 \end{pmatrix}.$$

Since the set of immersed flowers is connected and s's do not vanish the sign in this equation is the same for all flowers. Taking all the circles with the same radius one checks that the correct sign is plus, which implies the claim.

Given a hexagonal circle pattern it is convenient to associate its touching points as well as the cross-ratios s_k to the edges of the honeycomb lattice. Equation (5) is a partial difference equation on the honeycomb lattice. The cross-ratios on the edges of each hexagon satisfy (5). Moreover, it is easy to check that the He-Schramm equation is sufficient to guarantee the existence of the corresponding circle packing.

Proposition 3.2 *Given a positive-imaginary function $s : E \to i\mathbb{R}_+$ on the edges E of the honeycomb lattice satisfying (5) on each honeycomb, there exists unique (up to Möbius transformation) immersed hexagonal circle packing with the cross-ratios given by the corresponding values of s.*

Theorem 3.3 *A circle flower is conformally symmetric if and only if its opposite cross-ratios s_k are equal*

$$s_k = s_{k+3} \qquad (k \bmod 6). \tag{6}$$

Proof. The property (6) for conformally symmetric flowers follows from (i) of Theorem 2.2. A simple computation with the flowers in Fig.4 shows that the map (s_1, s_2) of immersed conformally

symmetric flowers to $(i\mathbb{R}_+)^2 \ni (s_1, s_2)$ is surjective. Since a flower is determined through the s's, the converse statement follows.

The general solution of (5, 6) on the whole HL depends on three arbitrary constants and can be given explicitly. There is a JAVA applet that lets you explore this three parameter family of circle packings interactively at [appl].

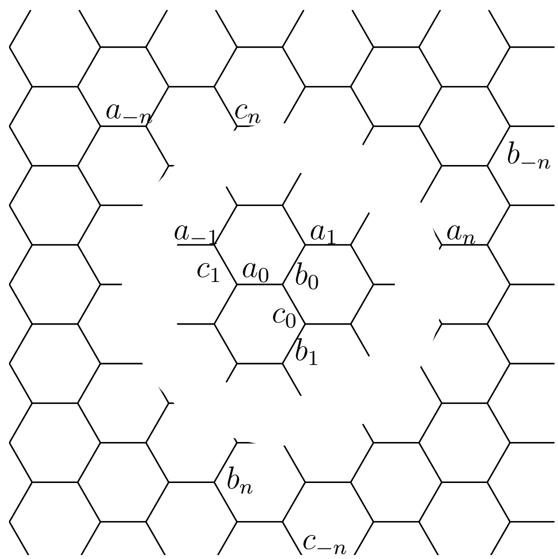

Figure 5: Cross-ratios of conformally symmetric circle patterns.

Theorem 3.4 *The general solution of (5, 6) is given by*

$$\begin{aligned} a_n &= i \tan(\Delta n + \alpha), \\ b_n &= i \tan(\Delta n + \beta), \\ c_n &= i \tan(\Delta n + \gamma), \end{aligned} \tag{7}$$

where $\Delta = -\alpha - \beta - \gamma$ *and the cross-ratios* s_k *on the edges of the hexagonal lattice are labeled by* a_n, b_n, c_n *as shown in Fig.5.*

Proof. We start with a simple proof of the consistency of the following continuation of a solution of (5, 6). Given s satisfying (5, 6) on a honeycomb H, i.e. a, b, c in Fig.6 satisfying

$$a + b + c + abc = 0, \tag{8}$$

and a value of s on one of the edges attached to the honeycomb (for example, d_1 in Fig.6), it can be uniquely extended to the full six honeycombs $H_1, \ldots, H_6$ neighboring H. Indeed, (5, 6) yield

$$b + d_1 + d_2 + bd_1d_2 = 0,$$

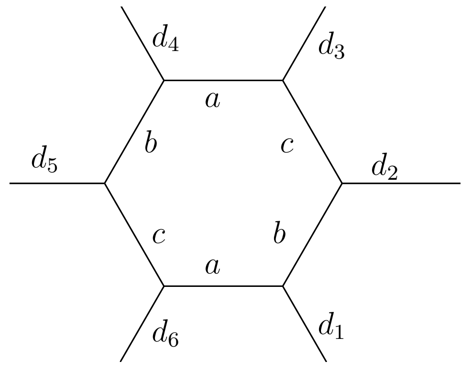

Figure 6: Continuation of conformally symmetric s about a honeycomb.

thus $d_2 = M_1(d_1)$ is a Möbius transformation of d_1. Passing once around the honeycomb H in this way one can check that (8) implies the monodromy Möbius transformation $M = M_6 \dots M_1$ is the identity, thus this continuation implies no constraints on d_1.

Proceeding this way, one reconstructs s on the whole lattice HL from its values on three adjacent edges (a, b, d_1 above). Then (5, 6) imply

$$a_n + b_{-n} + c_1 + a_n b_{-n} c_1 = 0, \qquad a_{n+1} + b_{-n} + c_0 + a_{n+1} b_{-n} c_0 = 0$$

and similar relations for other a_n, b_n, c_n. These identities become just the addition theorem for the tangent function, implying the formulas in (7), which can be checked directly.

4 The Doyle spirals

Denote by R the radius of the center circle of a flower and by $R_k,\ k = 1, \dots, 6$, the radii of its petals. The Doyle spirals are characterized through the constraint (see [BDS, CR] for a complete analysis of Doyle spirals)

$$R_k R_{k+3} = R^2, \qquad R_k R_{k+2} R_{k+4} = R^3 \tag{9}$$

on the radii of the circles (see Fig.7 where the central radius is normalized to be $R = 1$). The Doyle spirals have two degrees of freedom (for example the ratios R_1/R and R_2/R, which are the same for the whole spiral) up to similarities. Again, you can experiment with the two radii in a JAVA applet [appl].

Proposition 4.1 *The Doyle spirals are conformally symmetric.*

Proof. The configurations of four touching circles with the radii R, R_{k-1}, R_k, R_{k+1} and with the radii $R_{k+3}, R_{k+4}, R, R_{k+2}$ differ by scaling. This implies $s_k = s_{k+3}$ (use both (3) and the second representation of s_k in (4)) and the claim follows by Theorem 3.3.

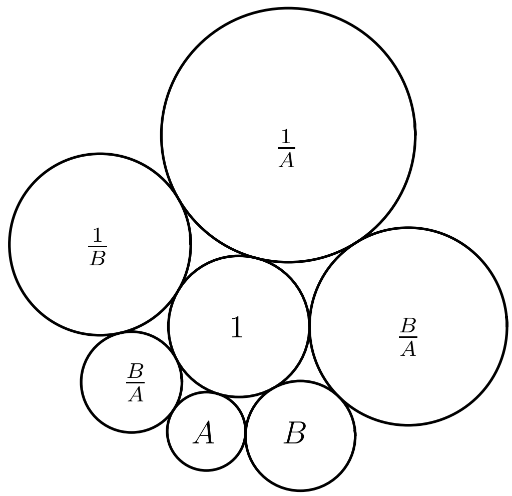

Figure 7: Radii of a Doyle spiral with the normalized central radius $R = 1$.

Theorem 4.2 *The Doyle spirals and their Möbius transformations can be characterized by the following two equivalent properties:*

(i) The circle packing is conformally symmetric, and the corresponding solution of (5) is "constant". It is of the form (7) with $\alpha, \beta, \gamma \in (0, \frac{\pi}{2})$, $\alpha + \beta + \gamma = 0 \pmod{\pi}$ *or, equivalently,*

$$a_n = a_0, \quad b_n = b_0, \quad c_n = c_0, \quad a_0, b_0, c_0 \in i\mathbb{R}_+, \quad a_0 + b_0 + c_0 + a_0 b_0 c_0 = 0.$$

(ii) The whole circle packing is invariant with respect to the Möbius involution of each of its flowers.

Proof. All the flowers of a Doyle spiral differ by scaling, which implies (i). Consider the Doyle spiral as in Fig.7. Computing the cross-ratios through the radii, one shows that the map

$$\{(A, B) \in \mathbb{R}_+^2\} \to \{(a, b, c) \in (i\mathbb{R}_+)^3 : a + b + c + abc = 0\}.$$

is surjective, thus (i) characterizes the Doyle spirals and their Möbius transforms. The proof of the equivalence $(i) \Leftrightarrow (ii)$ is elementary and is left to the reader.

It is an open problem whether the Doyle spirals are the only entire circle packings. Formulas (7) imply that it is possible to have all cross-ratios being positive imaginary (necessary condition for entireness) only when $\Delta = 0$.

Corollary 4.3 *Doyle spirals are the only entire conformally symmetric circle patterns.*

5 Airy functions as continuous limit

Because of the property (9), Doyle spirals are interpreted as a discrete exponential function.

In the conformal setting this interpretation can also be easily observed. Indeed, let P^ϵ be a family of circle packings approximating a holomorphic mapping in the limit $\epsilon \to 0$. In [HS] He and Schramm investigated the behavior of the cross-ratios s_k in this limit:

$$s_k = i\sqrt{3}(1 + \epsilon^2 h_k^\epsilon),$$

where h_k is called the discrete Schwarzian derivative (Schwarzian) of P^ϵ at the corresponding edge of the hexagonal lattice. The discrete Schwarzians converge to the Schwarzian derivative

$$S(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{10}$$

of the corresponding holomorphic mapping. More precisely, there exist continuous limits

$$a = \lim_{\epsilon \to 0} h_1^\epsilon, \quad b = \lim_{\epsilon \to 0} h_2^\epsilon, \quad c = \lim_{\epsilon \to 0} h_3^\epsilon$$

for the smooth functions[7] a, b, c. Because of (5) these functions satisfy

$$a + b + c = 0 \tag{11}$$

at each point. The Schwarzian equals

$$S(f) = 4(a + q^2 b + qc), \qquad q = e^{2\pi i/3},$$

and, using (11), this also yields

$$6a = \mathrm{Re}(S(f)), \quad 6b = \mathrm{Re}(qS(f)), \quad 6c = \mathrm{Re}(q^2 S(f)). \tag{12}$$

We see that, due to Theorem 4.2, Doyle spirals correspond to holomorphic functions with constant Schwarzian derivative $S(f) = \text{const}$. The general solution of the last equation is the exponential function and its Möbius transformations.

It is natural to ask which holomorphic functions correspond to general conformally symmetric circle packings. In Fig.5 one observes that each of the cross-ratios a_n, b_n, c_n is constant along one lattice direction. For the functions a, b, c above, this implies

$$a = a(\mathrm{Re}(z)), \qquad b = b(\mathrm{Re}(qz)), \qquad c = c(\mathrm{Re}(q^2 z)), \tag{13}$$

where z is the complex coordinate. Comparing (12) and (13) we see that the Schwarzian is a linear function of z:

$$S(f) = Az + B, \qquad A \in \mathbb{R},\ B \in \mathbb{C}. \tag{14}$$

[7]Note that $\lim_{\epsilon \to 0} h_k^\epsilon = \lim_{\epsilon \to 0} h_{k+3}^\epsilon$

Equation (14) can be easily solved by standard methods. The general solution of $S(f) = u(z)$ with holomorphic $u(z)$ is given by $f(z) := \psi_1/\psi_2$, where $\psi_1(z)$ and $\psi_2(z)$ are two independent solutions of the linear differential equation $\psi'' = u(z)\psi$.

By a shift and scaling of the variable z, equation (14) with $A \neq 0$ can be brought to the form

$$S(f) = z. \tag{15}$$

Solutions of the corresponding linear equation

$$\psi'' = z\psi \tag{16}$$

are the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. On the real line the first one is given by [SO]

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(xt + \frac{t^3}{3})dt,$$

and the second one is related to it by

$$\mathrm{Bi}(z) = iq^2\mathrm{Ai}(q^2 z) - iq\mathrm{Ai}(qz).$$

In the corresponding Möbius class of solutions of (15) it is natural to choose

$$f(z) := \frac{\mathrm{Bi}(z) - \sqrt{3}\mathrm{Ai}(z)}{\mathrm{Bi}(z) + \sqrt{3}\mathrm{Ai}(z)}, \tag{17}$$

which is the most symmetric one, $f(qz) = qf(z)$. The corresponding circle packing, symmetric with respect to the rotation $z \to qz$, is shown in Fig.8.

6 About the interactive version of this paper

Since the families of circle packings discussed in this paper have a finite (and even small) number of parameters it seemed to be natural to look for a way to visualize the whole families.

Except the Doyle spirals the families are only defined modulo an arbitrary Möbius transformation. Therefore it should be possible to view the packings with a Möbius transformation applied, so one could look at them through "Möbius glasses".

The outcome of this is an interactive version of this paper. This version includes some java applets that let you experiment with the circle packings directly. In particular there are applets to illustrate the families of circle packing flowers, the whole class of conformally symmetric circle packings, and the special case of Doyle spirals.

This interactive version renders a dvi file inside a java applet. It needs a web browser that includes a java vm. Since flipping pages might get slow on old machines and java vm's that have no just in time compiler, we also provide a page that shows the applets that are missing in this version only. You will find the paper and the applets at

http://www-sfb288.math.tu-berlin.de/Publications/online/

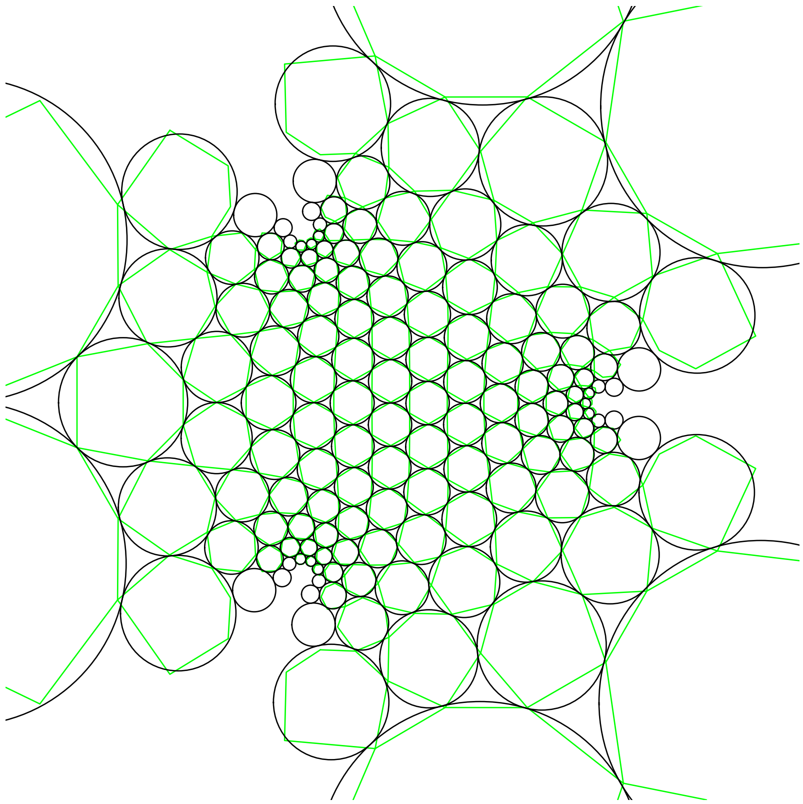

Figure 8: A conformally symmetric circle packing (with $\alpha = \beta = \gamma$ in (7)) and its smooth counterpart. The vertices of the hexagons are the images of the points of a standard hexagonal grid under the map f from (17).

References

[AB] S.I. Agafonov, A.I. Bobenko, Discrete z^γ and Painlevé equations, *Internat. Math. Res. Notices* **4** (2000).

[BDS] A.F. Beardon, T. Dubejko, K. Stephenson, Spiral hexagonal circle packings in the plane, *Geom. Dedicata* **49** (1994) 39-70.

[BS] A.F. Beardon, K. Stephenson, The Uniformization Theorem for Circle Packings, *Indiana Univ. Math. J.* **39:4** (1990) 1383-1425.

[BP] A.I. Bobenko, U. Pinkall, Discretization of surfaces and integrable systems, In: *Discrete Integrable Geometry and Physics*; Eds.: A.I. Bobenko and R. Seiler, Oxford University Press, 1999, 3-58.

[CR] K. Callahan, B. Rodin, Circle packing immersions form regularly exhaustible surfaces, *Complex Variables* **21** (1993) 171-177.

[DS] T. Dubejko, K. Stephenson, Circle Packings: Experiments in Discrete Alalytic Function Theory, *Experimental Math.* **4:4** (1995) 307-348.

[HS] Zh. He, O. Schramm, The C^∞-convergence of hexagonal disk packings to the Riemann map, *Acta Math.* **180** (1998) 219-245.

[R] B. Rodin, Schwarz's lemma for circle packings, *Invent. Math.* **89** (1987) 271-289.

[RS] B. Rodin, D. Sullivan, The convergence of circle packings to Riemann mapping, *J. Diff. Geometry* **26** (1987), 349-360.

[S] O. Schramm, Circle patterns with the combinatorics of the square grid, *Duke Math. J.* **86** (1997) 347-389.

[Swww] O. Schramm, WWW-home page, http://www.math.weizmann.ac.il/˜schramm/talks.

[SO] J. Spanier, K.B. Oldham, Atlas of functions, Hemisphere Publishing Corporation, Springer (1987).

[T] W.P. Thurston, The finite Riemann mapping theorem, Invited address, International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University (1985).

[appl] http://www-sfb288.math.tu-berlin.de/Publications/online/cscpOnline/Applets.html

[online] http://www-sfb288.math.tu-berlin.de/Publications/online/cscpOnline/index.html